10.1 Higher Sobolev Spaces

We know that $H^0(\mathbb{S}) = L^2(\mathbb{S})$. We can iteratively define $f \in H^k(\mathbb{S})$, $k \in \mathbb{N} \Leftrightarrow f \in H^{k-1}(\mathbb{S})$ and $\frac{d^{k-1}f}{dx^{k-1}} \in H^1(\mathbb{S})$.

So for $f \in H^k(\mathbb{S})$, define

$$L^2(\mathbb{S}) \ni \frac{d^k f}{dx^k} \equiv \frac{d}{dx}\frac{d^{k-1}f}{dx^{k-1}} \in L^2(\mathbb{S})$$

Theorem. *$f \in H^k(\mathbb{S})$ if and only if f has weak derivatives of all order $l \leq k$. That is $\exists h_l \in L^1(\mathbb{S})$ such that*

$$\int_{\mathbb{S}} f\frac{d^l\varphi}{dx^l} = (-1)^l \int_{\mathbb{S}} h_l\varphi, \qquad \forall\varphi \in C^\infty(\mathbb{S})$$

Proof.

$$\int f\frac{d^k\varphi}{dx^k}dx = \int f\frac{d^{k-1}}{dx^{k-1}}\left(\frac{d\varphi}{dx}\right)dx = (-1)^{k-1}\int \underbrace{h_{k-1}\frac{d\varphi}{dx}}_{dh^{k-1}/dx^{k-1}} = (-1)^k\int h_k\varphi$$

□

Theorem. $f \in H^k(\mathbb{S}) \Leftrightarrow f = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} c_n e^{inx}$ *and* $\sum_{n\in\mathbb{Z}} n^{2k}|c_n|^2 < \infty$

Proof. Just use induction. □

Definition. All these $H^k(\mathbb{S})$ are Hilbert spaces with the norms

$$\|f\|^2_{H^k} = \frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}}(1 + n^2 + n^4 \cdots + n^{2k})|c_n|^2$$

This is equivalent to the (Hilbert) Norm

$$\|f\|^2_{H^k} = \frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}}(1 + n^{2k})|c_n|^2 \equiv \frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}}(1 + n^2)^k|c_n|^2$$

This "makes sense" for $k \in [0, \infty)$.

Definition. For $r \in \mathbb{R}^+$ define $H^r(\mathbb{S}) \subset L^2(\mathbb{S})$ such that $f \in H^r(\mathbb{S})$ if $\exists f_p \to f$ in $L^2(\mathbb{S})$, $f_p \in C^\infty(\mathbb{S})$ and $\|f_p\|^2_{H^r} = \sum(1+n^2)^r|c_n(f_p)|^2 < \infty$ i.e. $f \in L^2(\mathbb{S})$ and Fourier series satisfies $\sum_{n\in\mathbb{Z}}(1+n^2)^r|c_n|^2 < \infty$.

Elementary Properties

- $H^{r'}(\mathbb{S}) \subset H^r(\mathbb{S})$ if $r' \geq r$.
- H^r has the inner product

$$\langle f, g\rangle_r = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}(1+n^2)^r c_n \overline{d_n}$$

- $f \in H^r(\mathbb{S})$ then

$$\sum_{n\in\mathbb{Z}}|(1+n^2)^{r/2}c_n|^2 < \infty$$

- Completeness (Left as exercise)

What this got to do with derivatives? $C^\infty(\mathbb{S}) \subset H^r(\mathbb{S})$ $\forall r$ but in fact

$$\bigcap_{r>0} H^r(\mathbb{S}) = C^\infty(\mathbb{S})$$

this is not too hard to prove. We have to show that

$$\bigcap_{r>0} H^r(\mathbb{S}) \subset C^\infty(\mathbb{S}) \qquad C^\infty(\mathbb{S}) \subset \bigcap_{r>0} H^r(\mathbb{S})$$

neither of which is too hard.

Exercise: Prove the two inclusions above.

11 Poisson Summation

This is really a correlation between fourier transforms and fourier series

For transforms:

$$\hat{f}(z) = \int_{\mathbb{R}} e^{-itz} f(t)dt, \qquad f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{itz}\hat{f}(z)dz$$

for series ($g : \mathbb{R} \to \mathbb{C}$ 2π-periodic, $g \in L^1([-\pi, \pi])$.

$$c_k = \int_{-\pi}^{\pi} g(t)e^{-itz}dt, \qquad g(t) = \frac{1}{2\pi}\sum_{n\in\mathbb{N}} c_k e^{itk}$$

If we assume g is C^∞, then we can integrate by parts in c_k:

$$k^p c_k = i^p \int_{-\pi}^{\pi} g(t)\frac{d^p}{dt^p}e^{-itk}dt = (-i)^p \int_{-\pi}^{\pi}\frac{d^p g}{dt^p}e^{-itk}dt \Longrightarrow |k^p c_k| < c_p, \quad \forall k$$

for some constant c_p. If g is 2π-periodic and C^∞ then $|c_k| < c_p/k^p$ $\forall p$ and $g(t) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} c_k e^{itk}$ converges uniformly with all derivatives.

Theorem. *If $f \in S(\mathbb{R})$ then*
$$g(t) = \sum_{k\in\mathbb{Z}} f(t+2\pi k)$$
converges with all derivatives to a 2π-periodic function and the fourier coefficients $c_k(g) = \hat{f}(k)$, $\forall k \in \mathbb{Z}$.

Proof. Formally
$$\frac{d^p g}{dt^p} = \sum_{k\in\mathbb{Z}} \underbrace{\frac{d^p f}{dt^p}}_{\in S(\mathbb{R})}(t+2\pi k)$$
So this converges if we can prove uniform convergence in the Schwartz functions. But we can prove this. We know that $\sup_{t\in\mathbb{R}} |t^2 f(t)| < C$. Shift this and get $\sup_{t\in\mathbb{R}} |(t+2\pi k)^2 f(t+2\pi k)| < C$. If $t \in [-\pi,\pi]$, $|k| > 2$, then $|t+2\pi k| \geq 2\pi|k-1|$. So $\sup_{t\in[-\pi,\pi]} |k-1|^2|f(t+2\pi k)| < C$ implies that $|f(t+2\pi k)| \leq \frac{C}{|k-1|^2}$.

Now the real meat of Poisson summation: What are the Fourier coefficients of g? Well, $c_k = \int_{-\pi}^{\pi} g(t)e^{-itk}dt$, since this is a uniformly convergent series we can write
$$c_k = \sum_{p\in\mathbb{Z}} \int_{-\pi}^{\pi} f(t+2\pi p)e^{-itk}dt$$
Let $T = t+2\pi p$, $t \in [-\pi,\pi]$ be a change of variables, then we have
$$c_k = \sum_{p\in\mathbb{Z}} \int_{-\pi+2\pi p}^{\pi+2\pi p} f(T)e^{-i(T-2\pi p)k}dT = \sum_{p\in\mathbb{Z}} \int_{-\pi+2\pi p}^{\pi+2\pi p} f(T)e^{-iTk}dT$$
as p runs through the integers, the integrals domain becomes all of $\mathbb{R}$ (the intervals $[-\pi+2\pi p, \pi+2\pi p]$ are a countable number of intervals that decompose $\mathbb{R}$). Thus
$$c_k = \int_{\mathbb{R}} f(T)e^{-iTk}dT = \hat{f}(k)$$
So we get
$$\sum_{k\in\mathbb{Z}} f(t+2\pi k) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \hat{f}(k)e^{-ikt}$$
□

Now we can do things like
$$g(0) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} c_k = \sum_{k\in\mathbb{Z}} \hat{f}(k)$$
Note: The actual **Poisson Summation Formula** is usually stated as
$$\boxed{\sum_{p\in\mathbb{Z}} f(2\pi p) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \hat{f}(k)}$$
In general the left side is a relation of classical geometry and the right is a statement of analysis.